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ISOPARAMETRIC FINITE ELEMENTS FOR THE 2-DIMENSIONAL RLW EQUATION

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Abstract

In this article, we present isoparametric finite elements for the numerical solution of the 2-dimensional regularized long wave (RLW) equation. A semi discrete scheme and a fully discrete scheme are presented and the accuracy of these schemes are measured by some error norms.

1. Introduction

RLW equation is a form of nonlinear long wave. In the study of the nonlinear diffusion wave, RLW equation occupies an important position for it can describe a lot of important physical phenomenons like shallow water wave and ion wave etc.

There have been many articles with finite difference method, the standard of the finite element method, mixed finite element method to study numerical solution of 1-dimensional equation. For 2-dimensional equation, [1] discussed its cnoidal wave solution and the uniqueness of

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the periodic solution to Cauchy problem. [4] studied its explicit exact solutions and a class of travelling wave solutions. But for its numerical analysis are rare.

Normally, finite element method is an effective method of solving differential equations, but for the curved edge area with enough smooth boundary, if use usually finite element can make error order will be not affected.

In this work, it is shown that the optimal estimates for the error are obtainable by using isoparametric finite element approximation.

The initial-boundary value problem of 2-dimensional RLW equation can be described as:

$$\begin{cases} u_t + \alpha u_{x_1} + \beta u_{x_2} + \gamma u u_{x_1} + \delta u u_{x_2} - \lambda u_{x_1 x_1 t} - \theta u_{x_2 x_2 t} = 0, (x, t) \in \Omega \times [0, T], & (1.1) \\ u(x, 0) = g(x), \qquad x \in \Omega, & (1.2) \\ u(x, t) = 0, \qquad (x, t) \in \partial\Omega \times [0, T], & (1.3) \end{cases}$$

where $\alpha, \beta, \gamma, \delta, \lambda, \theta$ are constants, and $\gamma \neq 0, \delta \neq 0, \lambda > 0, \theta > 0.$

2. Notation

Let X be a Banach space with norm $\| \cdot \|_X$ and $f(x, t) : [0, T] \to X$ be Lebesgue measurable, the following norms are defined:

$$\| f \|_{L_2(X)} = \left(\int_0^t \| f \|_X^2\right)^{\frac{1}{2}} \qquad \| f \|_{L_\infty(X)} = \sup_{0 \le t \le T} \| f \|_X.$$

For some integer $J \geq 1,$ let $\Delta t = T / J,$ and $t_j = j\Delta t,$ denote

$$u^j = u|_{t = t_j} \qquad \partial_t u^j = \frac{1}{\Delta t}(u^{j+1} - u^j).$$

The generalized solution of (1.1)-(1.3) is that: We seek a function $u(x, t) \in H_0^1(\Omega), 0 \leq t \leq T,$ such that

$$\begin{cases} (u_t, v) + a(u_t, v) = \displaystyle\sum_{i=1}^{2} (b_i(u) u_{x_i}, v), \quad \forall v \in H_0^1(\Omega), & (2.1) \\ (u(x, 0), v) = (g(x), v), \qquad \forall v \in H_0^1(\Omega), & (2.2) \end{cases}$$

where $(u, v) = \int_{\Omega} uvdx$, $a(u, v) = \lambda(u_{x_1 t}, v_{x_1}) + \theta(u_{x_2 t}, v_{x_2})$, $b_1(u) = \alpha + \gamma u$, $b_2(u) = \beta + \delta u$.

After partition Ω by the isoparametric elements of degree k, we get its approximate area $\Omega_\hbar = \bigcup_{K \in T_\hbar} K$, where K is a partition unit. Denote $\hbar = \sup_{K \in T_\hbar} \operatorname{diam}(K)$. Following [2], there exists a mapping $\varphi_\hbar : \Omega_\hbar \to \Omega$, its inversion is $\varphi_\hbar^{-1}$, then $\varphi_\hbar|_K : K \to \breve{K}$ and $\Omega = \bigcup_{K \in T_\hbar} \breve{K}$.

Define the regional interpolation operator π, let π_K be the interpolation operator of the unit K. Then

$$(\pi v)|_K = \pi_K(v|_K), \quad \forall K \in T_\hbar.$$

We obtain the finite element subspaces of $H^1(\Omega_\hbar)$

$$V_\hbar = \{v_\hbar : v_\hbar = \pi v, \ v \in C^0(\overline{\Omega})\},$$

and the finite element subspaces of $H^1(\Omega)$

$$\breve{V}_\hbar = \{\breve{v}_\hbar : \breve{v}_\hbar = v_\hbar \circ \varphi_\hbar^{-1}, \ v_\hbar \in V_\hbar\}.$$

Lemma 2.1 [3]**.** *Assume that* p, q, r *are functions defined on* $\Omega_\hbar$, *and* $\breve{p} = p \circ \varphi_\hbar^{-1}$, $\breve{r} = r \circ \varphi_\hbar^{-1}$, *if* $p, q, r \in L_2(\Omega_\hbar)$, *then*

$$|(\breve{p}, \breve{r}) - (q, r)_\hbar| \le c[\| p - q \|^2_{L_2(\Omega_\hbar)} + \hbar^{2k} \| p \|^2_{L_2(\Omega_\hbar)} + \| r \|^2_{L_2(\Omega_\hbar)}]. \quad (2.3)$$

Lemma 2.2 [2]**.** *Let* $0 \le m \le k + 1$, *then* $\| r \|_{H_m(\Omega_\hbar)}$ *is equivalent to* $\| r \circ \varphi_\hbar^{-1} \|_{H_m(\Omega)}$.

Following this lemma, we obtain:

Lemma 2.3. *Let* $\tilde{u}$ *be* $H^1(\Omega_\hbar)$ *projection of* u *on* $V_\hbar$, *i.e.,* $\tilde{u} \in V_\hbar$, $0 \le t \le T$, *satisfies*

$$(\nabla(\widetilde{u} - u \circ \varphi_\hbar), \nabla V)_\hbar + (\widetilde{u} - u \circ \varphi_\hbar, V)_\hbar = 0, \quad \forall V \in V_\hbar. \tag{2.4}$$

If $u \in L_\infty(H^{k+1}(\Omega))$, $u_t \in L_2(H^{k+1}(\Omega_\hbar))$, *then*

$$\| u - \widetilde{u} \circ \varphi_\hbar^{-1} \|_{L_\infty(H^1(\Omega))} + \| (u - \widetilde{u} \circ \varphi_\hbar^{-1})_t \|_{L_2(H^1(\Omega))} \le c\hbar^k. \tag{2.5}$$

3. The Semidiscrete Scheme and its Error Estimation

The semidiscrete finite elements approximation of (2.1), (2.2) is: $u_h(x, t) \in V_\hbar$ is sought such that for any $t \in [0, T]$ satisfies

$$\begin{cases} (u_{ht}, v)_\hbar + a_\hbar(u_{ht}, v) = \sum_{i=1}^{2} (b_i(u_h) u_{hx_i}, v)_\hbar, & \forall v \in V_\hbar, \quad (3.1) \\ (u_h(x, 0), v)_\hbar = (g(x), v)_\hbar, & \forall v \in V_\hbar. \quad (3.2) \end{cases}$$

By the equation, it follows that $b_i(x, u)$, $(i = 1, 2)$ is bounded, and meet Lipschitz condition about u. It is easily shown that (3.1), (3.2) possesses a unique solution.

Now, let $w(x, t) \in V_\hbar$, $t \in [0, T]$. Denote $u_h - u \circ \varphi_\hbar = \xi + \eta$, where $\xi = u_h - w$, $\eta = w - u \circ \varphi_\hbar$.

(2.1) may be written

$$(w_t, v)_\hbar + a_\hbar(w_t, v) = (w_t, v)_\hbar - (u_t, v \circ \varphi_\hbar^{-1}) + a_\hbar(w_t, v)$$

$$- a(u_t, v \circ \varphi_\hbar^{-1}) + \sum_{i=1}^{2} (b_i(u) u_{x_i}, v \circ \varphi_\hbar^{-1}).$$

Using (3.1) minus the above, we obtain

$$(\xi_t, v)_\hbar + a_\hbar(\xi_t, v)$$

$$= [(u_t, v \circ \varphi_\hbar^{-1}) - (w_t, v)_\hbar] + [a(u_t, v \circ \varphi_\hbar^{-1}) - a_\hbar(w_t, v)]$$

$$+ [\sum_{i=1}^{2}(b_i(u_h)u_{hx_i}, v)_\hbar - \sum_{i=1}^{2}(b_i(u)u_{x_i}, v \circ \varphi_\hbar^{-1}]$$

$$\triangleq E_1 + E_2 + E_3. \tag{3.3}$$

In (3.3), choose $v = \xi$, the left-hand side becomes

$$(\xi_t, v)_\hbar + a_\hbar(\xi_t, v) \ge \frac{1}{2}\frac{d}{dt}\| \xi \|_{L_2(\Omega_\hbar)}^2 + \frac{k_0}{2}\frac{d}{dt}\| \nabla\xi \|_{L_2(\Omega_\hbar)}^2,$$

where $k_0 = \min\{\lambda, \theta\}$.

Now we estimate the right-hand side of (3.3), using Lemma 2.1, we have

$$\begin{aligned}
E_1 &= (u_t, v \circ \varphi_\hbar^{-1}) - (w_t, v)_\hbar \\
&\le c[\| \eta_t \|_{L_2(\Omega_\hbar)}^2 + \hbar^{2k}\| u_t \circ \varphi_\hbar^{-1} \|_{L_2(\Omega_\hbar)}^2 + \| \xi \|_{L_2(\Omega_\hbar)}^2] \\
&\le c[\| \eta_t \|_{L_2(\Omega_\hbar)}^2 + \hbar^{2k} + \| \xi \|_{L_2(\Omega_\hbar)}^2],
\end{aligned}$$

$$\begin{aligned}
E_2 &= a(u_t, \xi \circ \varphi_\hbar^{-1}) - a_\hbar(w_t, \xi) \\
&= a(u_t, \xi \circ \varphi_\hbar^{-1}) - a_\hbar((u \circ \varphi_\hbar)_t, \xi) - a_\hbar(\eta_t, \xi) \\
&\le c[\| \eta_t \|_{H^1(\Omega_\hbar)}^2 + \hbar^{2k} + \| \xi \|_{H^1(\Omega_\hbar)}^2 + \| \nabla\xi \|_{L_2(\Omega_\hbar)}^2] + k_1\| \nabla\eta_t \|_{L_2(\Omega_\hbar)}^2,
\end{aligned}$$

$$\begin{aligned}
E_3 &= \sum_{i=1}^{2}(b_i(u_h)u_{hx_i}, \xi)_\hbar - \sum_{i=1}^{2}(b_i(u)u_{x_i}, \xi \circ \varphi_\hbar^{-1}) \\
&= \sum_{i=1}^{2}(b_i(u_h)u_{x_i}, \xi)_\hbar + \sum_{i=1}^{2}(b_i(u_h)\eta_{x_i}, \xi)_\hbar \\
&\quad + \sum_{i=1}^{2}(b_i(u_h)(u \circ \varphi_\hbar)_{x_i}, \xi)_\hbar - \sum_{i=1}^{2}(b_i(u)u_{x_i}, \xi \circ \varphi_\hbar^{-1}) \\
&\triangleq E_{31} + E_{32} + E_{33} + E_{34},
\end{aligned}$$

where $k_1 = \max\{\lambda, \theta\}$.

From [3], we know

$$E_{31} + E_{32} \le c[\| \nabla\xi \|^2_{L_2(\Omega_\hbar)} + \| \xi \|^2_{L_2(\Omega_\hbar)} + \| \eta \|^2_{H^1(\Omega_\hbar)}],$$

$$E_{33} + E_{34} \le c\sum_{i=1}^{2}[\|(b_i(u \circ \varphi_\hbar) - b_i(u_\hbar))(u \circ \varphi\hbar)_{x_i}\|^2_{L_2(\Omega_\hbar)}$$

$$+ \hbar^{2k}\| (b_i(u \circ \varphi_\hbar))(u \circ \varphi\hbar)_{x_i} \|^2_{L_2(\Omega_\hbar)}\| \xi \|^2_{L_2(\Omega_\hbar)}]$$

$$\le c[\| \eta \|^2_{L_2(\Omega_\hbar)} + \hbar^{2k} + \| \xi \|^2_{L_2(\Omega_\hbar)}].$$

From the above estimations, we get

$$\frac{1}{2}\frac{d}{dt}\| \xi \|^2_{L_2(\Omega_\hbar)} + \frac{k_0}{2}\frac{d}{dt}\| \nabla\xi \|^2_{L_2(\Omega_\hbar)} \le c[\| \eta_t \|^2_{H^1(\Omega_\hbar)} + \hbar^{2k} + \| \eta \|^2_{H^1(\Omega_\hbar)}]$$

$$+ c\| \xi \|^2_{H^1(\Omega_\hbar)} + c\| \nabla\xi \|^2_{L_2(\Omega_\hbar)}.$$

Choosing $k = \min\{\frac{1}{2}, \frac{k_0}{2}\}$, then

$$k\frac{d}{dt}\| \xi \|^2_{H^1(\Omega_\hbar)} \le c[\| \eta_t \|^2_{H^1(\Omega_\hbar)} + \hbar^{2k} + \| \eta \|^2_{H^1(\Omega_\hbar)}] + c\| \xi \|^2_{H^1(\Omega_\hbar)}.$$

Using Gronwall inequality, we obtain

$$\| \xi \|^2_{H^1(\Omega_\hbar)} \le c\xi(0) + c[\| \eta_t \|^2_{L_2(H^1(\Omega_\hbar))} + \hbar^{2k} + \| \eta \|^2_{L_2(H^1(\Omega_\hbar))}].$$

We choose $w = \tilde{u}$ be the solution of (2.4), from (3.3), we obtain:

Theorem 3.1. *Let u and* u_h *be the solution of* (2.1), (2.2) *and* (3.1), (3.2), *respectively,* $u \in L_\infty(H^{k+1}(\Omega))$, $u_t \in L_2(H^{k+1}(\Omega))$, *and* $\| \nabla u \|^2_{L_\infty(L_\infty(\Omega))}$, $\| \nabla u_t \|^2_{L_\infty(L_\infty(\Omega))}$ *are bounded, then we have the optimal error estimate*

$$\| u_h \circ \varphi_\hbar^{-1} - u \|_{L_2(H^1(\Omega))} \le c\hbar^k.$$

4. The Fully Discrete Scheme and its Error Estimation

The fully discrete finite elements of (2.1), (2.2) is: $\{U^j\}_{j=1}^J \in V_h$ is sought for any $t \in [0, T]$ such that

$$\begin{cases} (\partial_t U^j, v)_h + a_h(\partial_t U^j, v) = \sum_{i=1}^{2} (b_i(U^j) U_{x_i}^j, v)_h, & \forall v \in V_h, \quad (4.1) \\ (U^0, v)_h = (g(x), v)_h, & \forall v \in V_h. \quad (4.2) \end{cases}$$

For any $\{w^j\}_{j=1}^J \in V_h$, denote $U^j - u^j \circ \varphi_h = \xi^j + \eta^j$, where $\xi^j = U^j - w^j$, $\eta = w^j - u^j \circ \varphi_h$, from (2.1), (2.2) and (4.1), (4.2), we obtain

$$\begin{aligned} &(\partial_t \xi^j, v)_h + a_h(\partial_t \xi^j, v) \\ &\quad = [(\partial_t u^j, v \circ \varphi_h^{-1}) - (\partial_t w^j, v)_h] \\ &\qquad + (u_t^j - \partial_t u^j, v \circ \varphi_h^{-1}) + [a(u_t^j, v \circ \varphi_h^{-1}) - a_h(\partial_t U^j, v)] \\ &\qquad + [\sum_{i=1}^{2} (b_i(U^j) U_{x_i}^j, v)_h - \sum_{i=1}^{2} ((b_i(u) u_{x_i})^j, v \circ \varphi_h^{-1})] \\ &\quad \triangleq Q_1 + Q_2 + Q_3 + Q_4. \end{aligned}$$

Choose $v = \xi^{j+1}$, from the left-hand side of the above, we have

$$\begin{aligned} (\partial_t \xi^j, \xi^{j+1})_h + a_h(\partial_t \xi^j, \xi^{j+1}) \geq \frac{1}{2\Delta t} [&\| \xi^{j+1} \|_{L_2(\Omega_h)}^2 - \| \xi^j \|_{L_2(\Omega_h)}^2 \\ &+ k_0(\| \nabla \xi^{j+1} \|_{L_2(\Omega_h)}^2 - \| \nabla \xi^j \|_{L_2(\Omega_h)}^2)]. \end{aligned}$$

Now we estimate the right-hand side, respectively,

$$\begin{aligned} Q_1 &= (\partial_t u^j, \xi^{j+1} \circ \varphi_h^{-1}) - (\partial_t w^j, \xi^{j+1})_h \\ &\leq c[\| \partial_t (u \circ \varphi)^j - \partial_t w^j \|_{L_2(\Omega_h)}^2 + h^{2k} \| \partial_t (u \circ \varphi)^j \|_{L_2(\Omega_h)}^2 \end{aligned}$$

$$+ \| \xi^{j+1} \|^2_{L_2(\Omega_\hbar)}]$$

$$\leq c[\| \partial_t \eta \|^2_{L_2(\Omega_\hbar)} + \hbar^{2k} + \| \xi^{j+1} \|^2_{L_2(\Omega_\hbar)}],$$

$$Q_2 = (u_t^j - \partial_t u^j, \xi^{j+1} \circ \varphi_\hbar^{-1})$$

$$\leq \| u_t^j - \partial_t u^j \|^2_{L_2(\Omega_\hbar)} + \| \xi^{j+1} \|^2_{L_2(\Omega_\hbar)}$$

$$\leq c(\Delta t)^2 \| u_{tt} \|^2_{L_2(\Omega_\hbar)} + \| \xi^{j+1} \|^2_{L_2(\Omega_\hbar)},$$

$$Q_3 = a(u_t^j, \xi^{j+1} \circ \varphi_\hbar^{-1}) - a_\hbar(\partial_t U^j, \xi^{j+1})$$

$$\leq c[\| \eta_t^j \|^2_{H^1(\Omega_\hbar)} + \hbar^{2k} + (\Delta t)^2 + \| \xi^{j+1} \|^2_{L_2(\Omega_\hbar)} + \| \nabla \xi^{j+1} \|^2_{L_2(\Omega_\hbar)}],$$

$$Q_4 = \sum_{i=1}^{2} (b_i(U^j) U^j_{x_i}, \xi^{j+1})_\hbar - \sum_{i=1}^{2} ((b_i(u) u_{x_i})^j, \xi^{j+1} \circ \varphi_\hbar^{-1})$$

$$\leq c[\| \eta^j \|^2_{H^1(\Omega_\hbar)} + \hbar^{2k} + (\Delta t)^2 + \| \xi^{j+1} \|^2_{L_2(\Omega_\hbar)} + \| \xi^{j+1} \|^2_{H^1(\Omega_\hbar)}].$$

From the above estimations, we obtain

$$\frac{1}{2\Delta t}[\| \xi^{j+1} \|^2_{L_2(\Omega_\hbar)} - \| \xi^j \|^2_{L_2(\Omega_\hbar)} + k_0(\| \nabla \xi^{j+1} \|^2_{L_2(\Omega_\hbar)} - \| \nabla \xi^j \|^2_{L_2(\Omega_\hbar)})]$$

$$\leq c[\| \eta^j \|^2_{H^1(\Omega_\hbar)} + \hbar^{2k} + (\Delta t)^2 + \| \xi^{j+1} \|^2_{L_2(\Omega_\hbar)} + \| \xi^{j+1} \|^2_{H^1(\Omega_\hbar)}$$

$$+ \| \eta^j \|^2_{H^1(\Omega_\hbar)} + \| \eta_t^j \|^2_{H^1(\Omega_\hbar)}].$$

Multiplication by $2\Delta t$, and sum j from 0 to $N - 1$, choose Δt sufficiently small, we obtain

$$\| \xi^N \|^2_{H^1(\Omega_\hbar)} \leq c[\hbar^{2k} + (\Delta t)^2 + \| \xi^0 \|^2_{H^1(\Omega_\hbar)}] + c\Delta t \sum_{j=0}^{N-1} (\| \eta_t^j \|^2_{L_2(\Omega_\hbar)}$$

$$+ \| \eta^j \|^2_{H^1(\Omega_\hbar)} + \| \eta_t^j \|^2_{H^1(\Omega_\hbar)} + \| \xi^j \|^2_{H^1(\Omega_\hbar)}).$$

Choosing $w_j = \tilde{u}^j$ be the solution of (2.4), from the above and Lemmas 2.1 and 2.2, by discrete Gronwall inequality, we obtain:

Theorem 4.1. *Suppose that* $b_{x_i u}(x, u), b_{x_i uu}(x, u)$ *exist and be bounded,* u_t, u_{tt}, u_{ttx_i} *are continuous about t. If u and* $\{U^j\}_{j=1}^J$ *are solutions of* (2.1), (2.2) *and* (4.1), (4.2), *respectively,* $u \in L_\infty(H^{k+1}(\Omega))$, $u_t \in L_2(H^{k+1}(\Omega))$, *and* $\| \nabla u \|_{L_\infty(L_\infty(\Omega))}, \| \nabla u_t \|_{L_\infty(L_\infty(\Omega))}$ *be bounded, then we have the following optimal error estimate*:

$$\| U \circ \varphi_h^{-1} - u \|_{L_\infty(H^1(\Omega_h))} \leq c(h^k + \Delta t).$$

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